# MIDTERM: ALGEBRA I

Date: **13th September 2013**
The Total points is **115** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (4+4+6+6 points) When is a ring called a reduced ring? Define integral domain. Let $R$ be a ring. Show that if $R$ is an integral domain then it is reduced. Is the converse true?
(2) (5+7+8 points) State Eisenstein's criterion for irreducibility. Show that the polynomial $f(X,Y,Z) = X^2Y^2Z^2 + Z^4 - Y^3Z + X^2Y$ is an irreducible polynomial in $R = \mathbb{C}[X,Y,Z]$. Let $I$ be the ideal $(U - f(X,Y,Z))$ in $R[U]$. Show that $R/I$ is a UFD.
(3) (5+15 points) Define local ring. Show that the power series ring $R[[X]]$ is a local ring if $R$ is a local ring.
(4) (20 points) Let $A$ a ring. Let $S$ be a multiplicative subset of $A$. Show that the functor $S^{-1}A \otimes_A -$ is an exact functor from $A$-modules to $S^{-1}A$-modules.
(5) (5+15 points) Define projective module. Let $R$ be a local ring and $M$ be a finitely generated projective $R$-module then $M$ is a free $R$-module.
(6) (15 points) Let $0 \to A_1 \to^f A_2 \to^g A_3 \to 0$ and $0 \to B_1 \to^F B_2 \to^G B_3 \to 0$ be short exact sequences of $R$-modules for a ring $R$. Let $d_i : A_i \to B_i$ be $R$-module homomorphism for $i = 1,2,3$ such that $d_2 \circ f = F \circ d_1$ and $d_3 \circ g = G \circ d_2$. Show that if $d_1$ is an isomorphism then $\ker(d_2) \cong \ker(d_3)$ and $\operatorname{coker}(d_2) \cong \operatorname{coker}(d_3)$.